Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

# A Quantum Convolutional Neural Network for Phase Detection on the Toric Code


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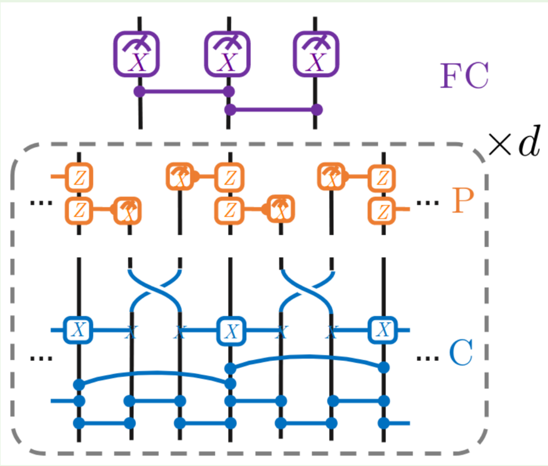
## Abstract

Quantum convolutional neural networks (QCNNs) are quantum circuits for recognizing quantum phases of matter at low sampling cost and have been designed for condensed matter systems in one dimension. Here we construct a QCNN that can perform phase recognition in two dimensions and correctly identify the phase transition from a toric code phase with $\mathbb{Z}_2$-topological order to the paramagnetic phase. The network also exhibits a noise threshold upto which the topological order is recognized. Our work generalizes phase recognition with QCNNs to higher spatial dimensions and intrinsic topological order, where exploration and characterization via classical numerics becomes challenging.


## Quantum Convolutional Neural Networks (QCNNs)

- Quantum circuit, inspired by classical convolutional neural networks, Renormalization Group (RG) flow and the Multiscale Entanglement Renormalization Ansatz (MERA) [1].




Schematic of QCNN circuit from [1]. Time propagates from the bottom to the top: convolutional layer (blue), pooling layer (orange) and fully connected output layer (purple). The convolutional and pooling layers can be repeated to increase output confidence.

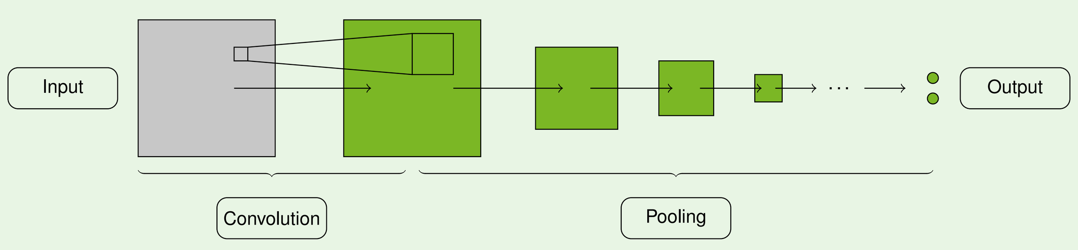
- Layers of iterable unitaries

$$U_{\mathrm{CP}}^{(j)} = U_{\mathrm{P}}^{(j)} U_{\mathrm{C}}^{(j)} \left(U_{\mathrm{C}}^{(j-1)}\right)^\dagger .$$

such that the full QCNN can be represented as

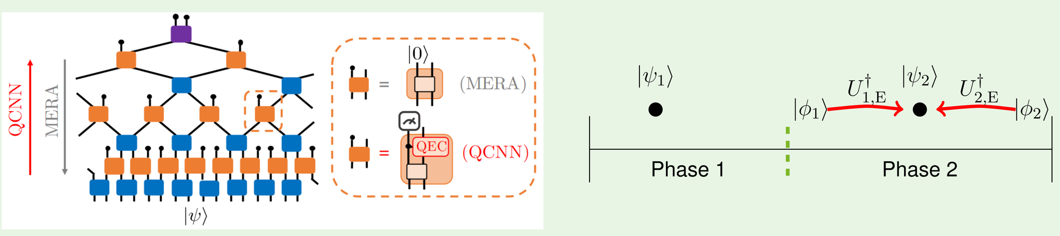
$$U_{\mathrm{QCNN}} = U_{\mathrm{FC}} \prod_j^d U_{\mathrm{CP}}^{(j)}$$

- Modification: Shortening QCNN to constant depth in the number of qubits by cancellation of subsequent convolutions




- QCNN removes short-range information and maps large-scale information to output qubits
- QCNN for phase recognition yields **improvement of sampling complexity** at phase boundary compared to direct measurement [2]

## Inspiration by MERA and RG Flow



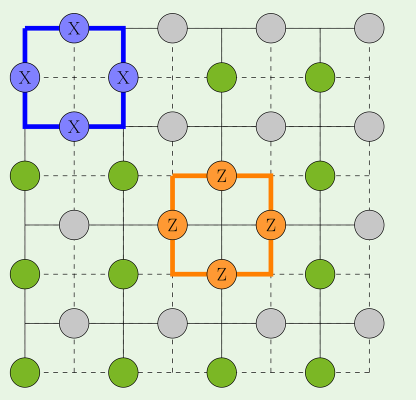

- **Multiscale Entanglement Renormalization Ansatz (MERA)** (left, taken from [1]):
  - Tree-structured quantum circuit
  - Generates many-body state by entangling additional qubits in a layer structure
- **Renormalization Group (RG)** flow (right) [3]:
  - Goal: finding phase of input state $|\phi_i\rangle$
  - Procedure:
    - Define reference states $|\psi_i\rangle$ for corresponding topological phases $\mathrm{SPT}_i$
    - States of same phase can be characterized as reference state with local unitary perturbations $|\phi_i\rangle = U_{i,\mathrm{E}}\,|\psi_i\rangle$
    - Phase transition cannot be crossed using only local unitaries
    - Implement error correction, that removes unitaries $U_{i,\mathrm{E}}$
    - Recover reference state of the corresponding phase

## Toric Code

- **2-dimensional periodic qubit lattice** with Hamiltonian

$$H_{\mathrm{TC}} = -\sum_i A_i - \sum_i B_i$$

with plaquette operator (blue) $A_i = \prod_{i=1}^4 \sigma_i^z$ and vertex operator (orange) $B_i = \prod_{i=1}^4 \sigma_i^z$



- Qubits displayed as gray and green dots
- Toric code in **magnetic field**

$$H_{\mathrm{mag}} = H_{\mathrm{TC}} - h_Z \sum_i^N \sigma_i^z - h_X \sum_i^N \sigma_i^x$$

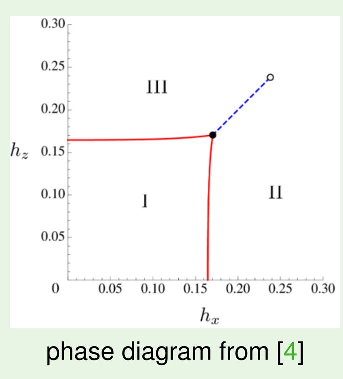
- **Single-qubit Pauli noise channel** in the form of

$$\rho = \mathcal{E}(|\psi\rangle\langle\psi|) = \sum_{l=0}^m K_l\,|\psi\rangle\langle\psi|\,K_l^\dagger$$

with $K_l \in \{\sqrt{p_{\mathrm{I}}}\mathbb{1},\ \sqrt{p_X}X,\ \sqrt{p_Z}Z\}^{\otimes N}$

## Phase Transition

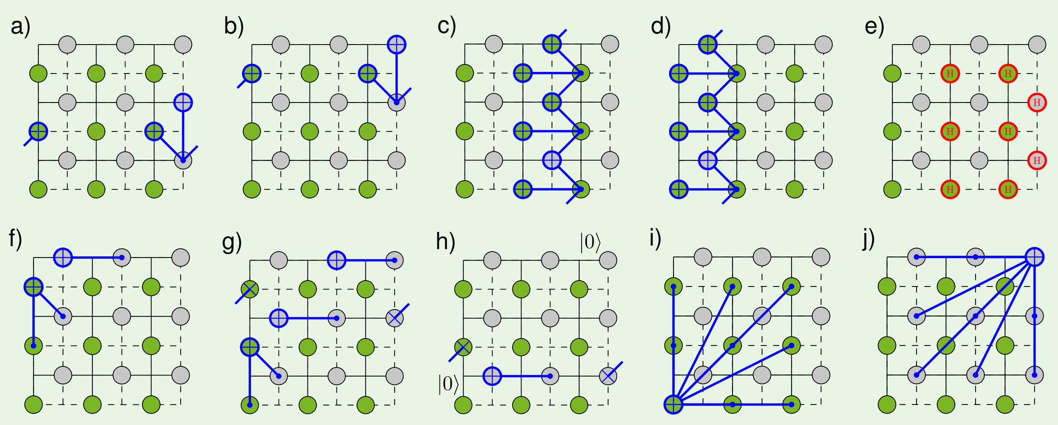
- Toric code undergoes a **phase transition in parallel magnetic field** [4]
- Critical field strength for magnetic field in Z-direction: $h_Z = 0.3284$
- **Goal:** observing threshold with reduced sample complexity via measurement of QCNN output




phase diagram from [4]

## Convolution

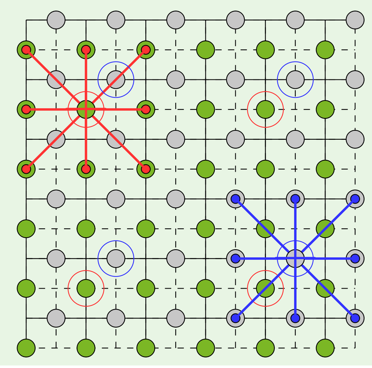
- Need to find circuit, which maps stabilizer measurements to qubits:
  - **Vertex** stabilizers are mapped to **gray (horizontal)** qubits
  - **Plaquette** stabilizers are mapped to **green (vertical)** qubits
- Starting point: adjoint ground state preparation circuit (a-e)
- Apply additional CNOTs for correct mapping of stabilizers (f, g)



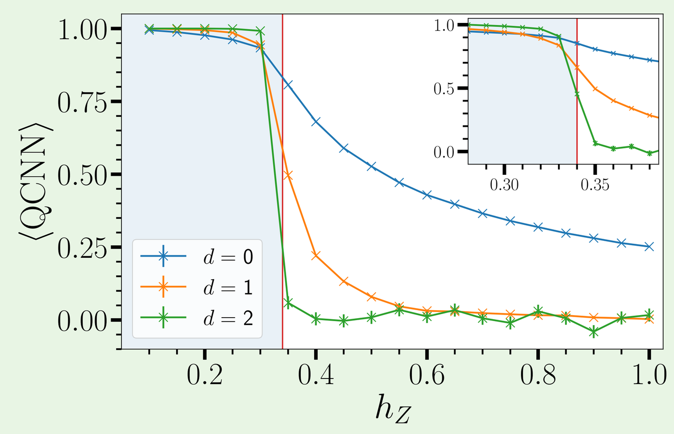

## Pooling

- **Local error correction** (LEC) on set of target qubits
- Goal: mimicking RG flow and recovering pure toric code as reference state on target qubits
- Chosen qubits will be propagated to next layer
- Pooling circuit:
  - Choose set of target qubits $t_j$ on vertical lattice edges (red circle) and horizontal lattice edges (blue circle)
  - Target qubits must be spaced out, such that the nearest neighbors do not overlap
  - Call nearest neighbors control qubits $c_{i,j} \in \mathcal{N}(t_j)$
  - Apply CNOTs between control qubits and target qubits
  - Apply Toffolis between nearest neighbors of control qubits $n_{k,i,j} \in \mathcal{N}(c_{i,j})$, controls qubits and target qubits

$$U_{P_l} = \left(\prod_{c\in\mathcal{N}(t)} \prod_{k\in\mathcal{N}(c)\setminus t} \mathrm{Toff}_{kct}\mathrm{CNOT}_{ct}\right)$$



## MPS Simulations for Ground States in Magnetic Field




Full simulated range (left), higher sample resolution around transition (right), Phase transition driven by Hamiltonian, Topologically ordered phase → Paramagnetic phase

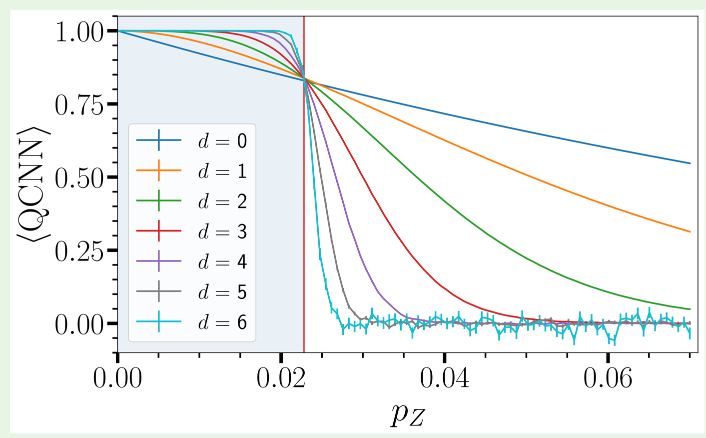
- **QCNN output:**

$$M_{\mathrm{QCNN}}^l = \frac{2}{N_l}\left(\sum_{j=1}^{N_l} m_j\right) - 1$$

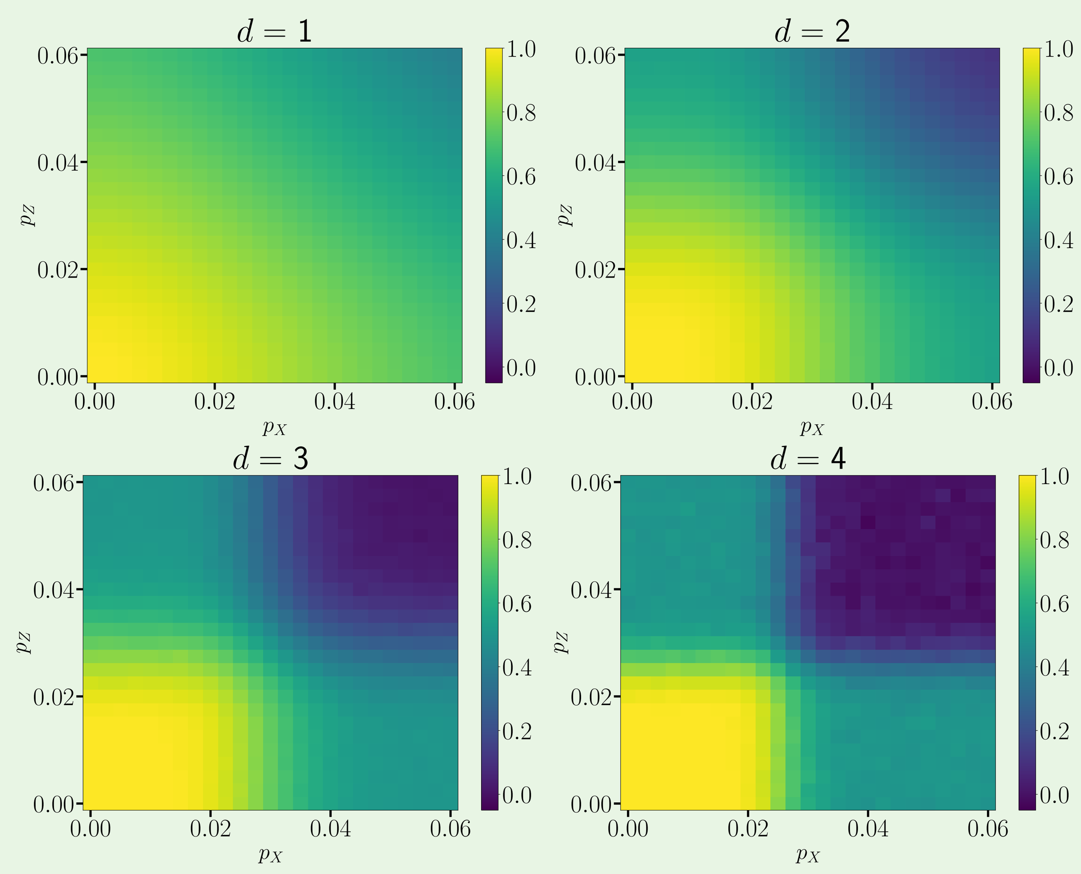
$$\langle M_{\mathrm{QCNN}}^l\rangle = \begin{cases} 1 & \text{if topol. phase} \\ 0 & \text{if paramagnetic/disordered} \end{cases}.$$

- with $N_l$: number of target qubits on lattice and $m_j$: outcome of Z-measurement on target-qubit $j$
- Simulation via Matrix Product States (MPS) on 2D-lattice
- Number of qubits $N_q = 486$
- Increase strength of magnetic field $h_Z$
- QCNN identifies transition correctly
- Increased step-likeness with additional pooling layers

## Simulations for Incoherent Pauli Noise

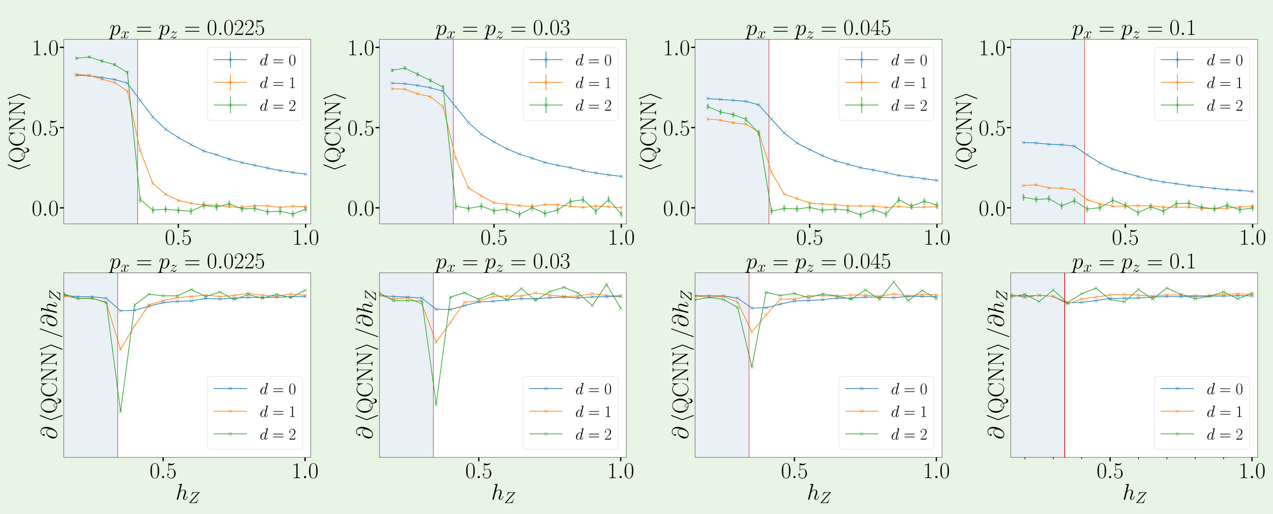



Transition on toric code driven by incoherent Pauli noise, Topologically ordered phase → Disordered regime




- Simulations for Pauli noise **classically computed via the evolution of toric code stabilizers under errors**
- Number of qubits $N_q \approx 9.6 \times 10^6$
- Transition in QCNN output at Pauli noise rate $p_Z = 2.28\%$
- Transition can be interpreted as a **decoder threshold** of the local error correction in pooling
- Simultaneous Pauli noise for in X- and Z-basis can be treated on the two separate sublattices at the same time (2D plots)
- Threshold simultaneously holds for both basis → square area (yellow) characterized as topological

## Phase Recognition in the Presence of Pauli Noise




- Combination of **ground states of the Hamiltonian with Pauli noise**
- Variation of the strength of the magnetic field in Z-direction $h_Z$ for simultaneous Pauli-X and Pauli-Z error rates $p_X = p_Z$
- Phase recognition for the transition from the topological to the paramagnetic phase **functional up to Pauli error threshold** $p = 2.28\%$
- After surpassing the Pauli threshold the pooling introduces additional errors that dampen the QCNN output
- **Density of errors increases** with each subsequent pooling layer
  → Output curve of higher layer drops below the one of the previous layer

## Conclusion and Outlook

**Summary:**

- **Successful construction of QCNN for 2D toric code** with phase recognition
- Application for lattice perturbation by random Pauli noise and magnetic field
- QCNN identifies phase transition **by design, no training required**
- Implementation as quantum circuit or classical post-processing

**Outlook:**

- Improvements for QCNN pooling?
- Increase error threshold for random Pauli noise?
- Improvements by training of parameterized circuit?
- Implementation on Hardware for small system?
- Use 2D tensor network to generate test data?

## References


[1] I. Cong et al. In: *Nature Physics* 15.12 (Dec. 2019), pp. 1273–1278.
[2] J. Herrmann et al. In: *Nature Communications* 13.1 (July 2022), p. 4144.
[3] E. Lake et al. Dec. 2022.
[4] J. Vidal et al. In: *Physical Review B* 79.3 (Jan. 2009), p. 033109.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